Numerical Solution of Ordinary Differential Equations

Differential Equations (DE)

Most of the real time physical systems are expressed in terms of rate of change. The Mathematical models that describe the state of such systems are often expressed in terms of not only certain system parameters but also their derivatives. Such model which contains functions and one or more of its derivatives is known as differential equation (DE). Each of the following equations are DE since they contain the derivative(s) of an unknown function y (a function of x) or z (a function of x and y).

i) $\frac{dy}{dx} + 3y = 2$

derivatives

function y (a function of x)

ii) $(\frac{dy}{dx})^2 + 2y^2 = 4(\frac{dy}{dx}) + 2x$

iii) $\frac{d3y}{dx3} + \frac{d2y}{dx2} + 8\frac{dy}{dx} - 9y = 10$

iv) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

Types of Differential Equations:

1) Ordinary Differential Equations

The equation which involves one or more ordinary derivatives of unknown of any order is called *Ordinary Differential Equation* (ODE).

Examples ODE are:

a) **Law of Cooling: dT(t)/dx = K[Ts-T(t)]** : The rate of loss of heat from a liquid is proportional to the difference of temperature between the liquid and the surroundings.

b) **Law of Motion**: mdv(t)/dt = F : The time rate change of velocity of a moving body is proportion to the force exerted by the body.

c) **Kirchhoff's Law for and Electric Circuit**: L di/dt = iR = V: The voltage across an electric circuit containing an inductance L and a resistance R.

The general form of the ordinary differential equation is: F (x, y, y', y'', … y^n) = 0. Where, 'x' is independent variable, y is dependent and y', y'' … are derivatives of 'y' with respect to 'x'.

2) Partial Differential Equations

The equation which contains more than one independent variables, dependent variables and its partial derivative is called *Partial Differential Equation* (PDE).

Examples of PDE are:
a) **Heat Flow in a Rectangular Plate:**

$$\frac{d^2y}{dx^2} + \frac{d^2y}{dz^2} = f(x, y)$$

Order and Degree of a Differential Equation

The ***order*** of a differential equation is its highest derivative. If the equation only contains a first derivative, it is called a differential equation of the first order. By derivative order we mean how many times the function was differentiated.

- A first order DE can be expressed in the form $\frac{dy}{dx} = f(x, y)$ and a second order DE can be expressed in the form $y'' = f(x, y, y')$.

Example:

$$\frac{dy}{dx} + 3y = 2 \text{--} (1)$$

Equation (1) has only first derivatives $\frac{dy}{dx}$ so it is first order derivatives.

$$\left(\frac{dy}{dx}\right)^2 + 2y^2 = 4\left(\frac{dy}{dx}\right) + 2x \text{ ------------------------------} (2)$$

Equation (2) has second derivatives $\frac{d^2y}{dx^2}$ so it is second order derivatives.

The ***degree*** is the exponent of the highest derivatives.
Example:

$$\left(\frac{dy}{dx}\right)^2 + 2y^2 = 4\left(\frac{dy}{dx}\right) + 2x \text{ ------------------------------} (2)$$

Equation (2) has highest derivative of dy/dx, and it has an exponent of 2, so this is "Second Degree"

.

Linear and Nonlinear Differential Equation

A DE is known as a *linear* DE when it does not contain terms involving the products of the dependent variable or its derivatives. For example, $y'' + 3y' = 2y + x^2$ is a second order, linear DE. The equations i) $y'' + (y')^2 = 1$ ii) $y' = -ay^2$ are *nonlinear* because the first one contains a product of y' and the second contains a product of y.

General and Exact Solution of DE

A solution to a DE is a relationship between the dependent and independent variable that satisfy the DE. For example, $y = 3x^2 + x$ is the solution of $y' = 6x + 1$. Also, $y = 3x^2+x+2$ is a solution of $y' = 6x + 1$. Actually, there is infinite number of solutions. In general, $y' = 6x + 1$ has a solution of the form $y = 3x^2 + x + c$ where c is known as the constant of integration. The solution that contains arbitrary constants is not unique and is therefore known as the *general solution*. If the values of the constants are known, on substitution of these values in the general solution, a unique solution known as *exact solution* can be obtained.

Initial Conditions

In order to obtain the values of the integration constants, we need additional information. If the order of the equation is n, we will have to obtain n constants and, therefore, we need n conditions in order to obtain an exact solution. These conditions are called *initial conditions*. For example, if the initial condition is $y(0) = 2$ then $c = 2$ and hence $y = 3x^2 + x + 2$ is the exact solution of $y' = 6x + 1$.

Initial Value & Boundary Value Problem

When all the conditions are specified at a particular value of the independent variable x, then the problem is called an *initial value problem*. If the initial conditions are specified at different values of the independent variable, then such problem is called *boundary value problem*.

Taylor's Series Method to solve ODE

Let $y = f(x)$, be a solution of the equation

$$\frac{dy}{dx} = f(x, y) \text{ where } y(x_0) = y_0$$

Expanding the Taylor's series about the point x_0, we get

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \ldots$$

$$y = f(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \ldots$$

Putting $x = x_1 = x_0 + h$, we get

$$y_1 = f(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

where $h = x_1 - x_0$.

Similarly,

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \ldots$$

The above formula is known as *Taylor's series method*.

- The major problem with the Taylor's series method is the evaluation of higher-order derivatives. They become very complicated. This method is, therefore, generally impractical from a computational point of view. However, it illustrate the basic approach to numerical solution of DE.

Example: Solve $\frac{dy}{dx}$ = x + y, y (1) = 0, numerically up to x = 1.2, with h = 0.1.

Solution: Here we have $x_0 = 1$, $y_0 = 0$ and

$\frac{dy}{dx} = y' = x + y \quad \Rightarrow y_0' = 1 + 0 = 1$

$y'' = 1 + y' \quad \Rightarrow y_0'' = 1 + 1 = 2$

$y''' = y'' \quad \Rightarrow y_0''' = 2$

$y^{iv} = y''' \quad \Rightarrow y^{iv} = 2$

Substitute the above values in the Taylor's series method,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$$

$$= 0 + 0.1/1 * 1 + (0.1)^2/2 * 2 + (0.1)^3/6 * 2 + (0.1)^4/24 * 2 + \ldots$$

$$= 0.11033846$$

$$\therefore y_1 = y(1.1) \approx 0.110$$

Now, $x_1 = x_0 + h = 1 + 0.1 = 1.1$ and $y_1 = 0.11$

We have,

$y_1' = x_1 + y_1 = 1.1 + 0.11 = 1.21$

$y_1'' = 1 + y_1' = 1 + 1.21 = 2.21$

$y_1''' = y_1'' = 2.21$

$y_1^{iv} = y_1''' = 2.21$

Substitute the above values in the Taylor's series method,

$$y_2 = y_0 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots$$

$$= 0.11 + 0.1/1 * 1.21 + (0.1)^2/2 * 2.21 + (0.1)^3/6 * 2.21 + (0.1)^4/24 * 2.21$$

$$= 0.232 \text{ (approximately)}$$

$$\therefore y_2 = y(1.2) \approx 0.232$$

One Step and Multi Step Method of Solution:

In one step methods, we use information from only one preceding point, i.e. to estimate the value of y_i, we need the conditions at the previous point y_{i-1} only. Multistep methods use information at two or more previous steps to estimate a value.

Euler's Method

The solution of differential equation by Taylor's Series Method gives the equation in the form of power series. We will now discuss the methods which give the solution in the form of a set of tabulated values. Euler's Method is one of the simplest one-step methods and it has limited application because of its low accuracy.
Eular's method is a numerical technique that solve ordinary differential equations in the form of : $\frac{dy}{dx}=f(x,y), y(0) = y0$

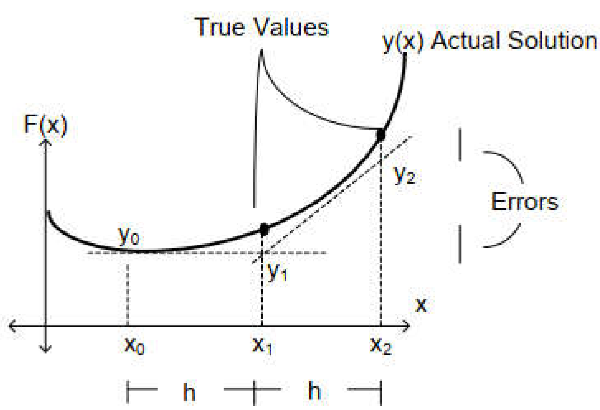

Fig: Illustration of Euler's Method

Only first order derivatives can be solved by Euler's Methods.

Derivation of Eular's Methods:

Consider the Taylor's series method

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \ldots + \frac{(x-x_0)^n}{n!} y^n(x_0)$$

Taking the first two terms,

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0)$$

Given the Differential Equation

$$y'(x_0) = f(x, y) \text{ with } y(x_0) = y_0$$

We have,

$$y'(x_0) = f(x_0, y_0)$$

$f(x_0, y_0) = \frac{y(x)-y(x_0)}{x-x_0}$

$\therefore\ y(x) = y(x_0) + (x - x_0) f(x_0, y_0)$

Then, the value of y(x) at $x = x_1$ is given by

$y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$

Letting step size $h = x_1 - x_0$, we obtain

$y_1 = y_0 + h f(x_0, y_0)$

Similarly,

$y_2 = y_1 + h f(x_1, y_1)$

In general form,

$y_{i+1} = y_i + h f(x_i, y_i)$ ---(1)

The above formula is known as *Euler's method* and can be used recursively to evaluate y_1, y_2, …. of $y(x_1)$, $y(x_2)$, … starting from the initial condition $y_0 = y(x_0)$.

- Euler's method is the simplest method and has a limited application because of its low accuracy.
- In this method the new value is obtained by extrapolating linearly over the step size h using the slope at its previous value i.e. New value = Old value + Slope * Step size.

Example: Solve the equation $\frac{dy}{dx} = 1 - y$, with the initial condition x = 0, y = 0, using Euler's method and tabulate the solutions at x = 0.1, 0.2, 0.3

Solution: Here, $f(x, y) = 1 - y$

Here,

$h = 0.1$

$x_0 = 0,\ y_0 = 0$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = 0.2$

$x_3 = 0.3$

Taking n = 0 in

$y_{n+1} = y_n + h f(x_n, y_n)$

We get,

$y_1 = y_0 + h f(x_0, y_0)$

$= 0 + (0.1)(1 - 0)$

$= 0.1$

Now, $y_2 = y_1 + h f(x_1, y_1)$

$= 0.1 + (0.1)(1 - 0.1)$

$= 0.19$

$y_3 = y_2 + h f(x_2, y_2)$

$= 0.19 + (0.1)(1 - 0.19)$

$= 0.271$

Example: Use Euler's Method with h=0.1 to solve $\frac{dy}{dx}$ = $x^2 + y^2$ with y(0) = 0 in the range 0<=x<=0.05

Solution:
Here, $f(x, y) = x^2 + y^2$
$x_0 = 0$
$y_0 = 0$

Taking n = 0 in

$$y_{n+1} = y_n + h f(x_n, y_n)$$

We get,

$$\begin{aligned}y_1 &= y_0 + h f(x_0, y_0)\\ &= 0 + 0.1\,(x_0^2 + y_0^2) = 0 + (0.1)\,(0 + 0)\\ &= 0\end{aligned}$$

Now,

$$\begin{aligned}y_2 &= y_1 + h f(x_1, y_1)\\ &= 0 + (0.1)\,((0.1)^2 + 0)\\ &= 0.001\end{aligned}$$

$$\begin{aligned}y_3 &= y_2 + h f(x_2, y_2)\\ &= 0.001 + (0.1)\,((0.2)^2 + (0.001)^2)\\ &= 0.005\end{aligned}$$

$$\begin{aligned}y_4 &= y_3 + h f(x_3, y_3)\\ &= 0.005 + (0.1)\,((0.3)^2 + (0.005)^2)\\ &= 0.014\end{aligned}$$

$$\begin{aligned}y_5 &= y_4 + h f(x_4, y_4)\\ &= 0.014 + (0.1)\,((0.4)^2 + (0.014)^2)\\ &= 0.03002\end{aligned}$$

So,

$y(0) = 0$	$y(0.1) = 0$	$y(0.2) = 0.001$
$y(0.3) = 0.005$	$y(0.4) = 0.014$	$y(0.5) = 0.0300196.$

Error Analysis of Euler's Methods
The numerical solution for ODEs involves two types of truncation errors:
1. Local Truncation Error: Results from an application of the method in question over a single step.
2. Propagated Truncation Error: Results from the approximations produced during previous step.
3. Local Truncation Error + Propagated Truncation Error = Global Truncation Error

Error analysis of Euler's methods can be derived from Taylor's series.

Consider, $\frac{dy}{dx} = f(x, y) = y'$, ---(2)

where x is independent and y is dependent variables.

Equation 2 can be represented by Taylor series Expansion about a stating value (x_i, y_i)

$$y_{i+1} = y_i + \frac{h}{1!} y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \ldots + \frac{h^n}{n!} y_i^n + R^n \text{ -------------------------------------(3)}$$

Where, $h = x_{i+1} - x_i$ and $R^n = \frac{y^{n+1}}{(n+1)!}(\xi)$ = the reminder term, -------------------------------(4)

where ξ is in between x_{i+1} and x_i.

Substituting Equation 2 and 4 into equation 3 we get,

$$y_{i+1} = yi + f(xi, yi)\, h + \frac{f'}{2!}(xi, yi)\, h^2 + \text{----------------} + \frac{f^{n-1}}{n!}(xi, yi)\, h^n + O(h^{(n+1)}) \text{---------(5)}$$

$O(h^{(n+1)})$ is local truncation error and proportional to step size raised to the power $(n + 1)^{th}$ power.

$$y_{i+1} = y_i + h f(x_i, y_i) \text{ ---(1)}$$

Subtracting equation 1 with equation 5, we get:

Local Truncation error = $E_t = \frac{f'}{2!}(xi, yi)\, h^2 +$ ----------------------------$+ O(h^{(n+1)}$---------(6)

For small h we can ignore the $O(h^{(n+1)})$ as error decreases with order increases.

$$E_a = \frac{f'}{2!}(xi, yi)\, h^2$$

Or, $E_a = O(h^2)$, E_a is approximate local truncation error.

Heun's method

Since, Euler's Method does not require any differentiation and is easy to implement on computers. However, it major weakness is large truncation errors. This is due to its linear characteristic because it uses only the first two terms of the Taylor's Series. So Heun's Method is considered to be an improvement to Euler's Method.

Huen's Methods works as Predictor Corrector Approach. The basic principle of Predictor Corrector Approach is:

1. Predict a solution of given ODE
2. Correct the predictor equation

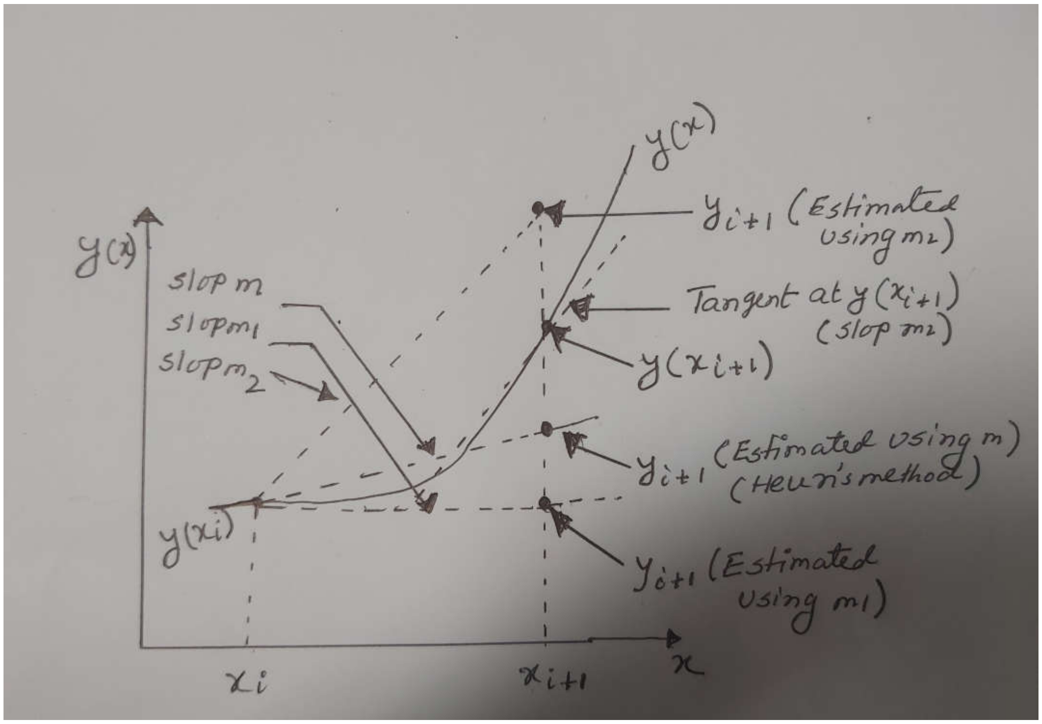

Fig 1: Illustration of Huen's Method

Derivation of Heun's Method:

In Euler's method, the slope at the beginning of the interval is used to extrapolate y_i to y_{i+1} over the entire interval. Thus, $y_{i+1} = y_i + m_1 h$ where m_1 is the slope at (x_i, y_i). As shown in following Fig 1, y_{i+1} is clearly an underestimate of $y\ (x_{i+1})$. This approach is known as predictor approach.

$$y_{i+1} = y_i + m_1 h$$

Predictor Equation

An alternative is to use the line which is parallel to the tangent at this point $(x_{i+1}, y\ (x_{i+1}))$ to extrapolate from y_i to y_{i+1}, as shown in Fig 1. i.e. $y_{i+1} = y_i + m_2h$ where m_2 is the slope at $(x_{i+1}, y\ (x_{i+1}))$. Note that the estimate appears to be overestimated.

A third approach is to use a line whose slope is the average of the slopes at the end points of the interval. Then $y_{i+1} = y_i + ((m_1+ m_2)/2)h$. As shown in Fig, this gives a better approximation to y_{i+1}. This approach is known as *Corrector Approach*.

$$y_{i+1} = y_i + ((m_1+ m_2)/2)h$$

Corrector Equation

The formula for implementing Heun's method can be constructed easily. Given the equation, $y'(x) = f(x, y)$, we can obtain

$$m_1 = y'(x_i) = f(x_i, y_i)$$
$$m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$$

therefore, $m = (f(x_i, y_i) + f(x_{i+1}, y_{i+1})) / 2$

So, $y_{i+1} = y_i + h/2\,[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$ ……. (1)

But the term y_{i+1} appears on both sides of Eq. (1) and therefore, y_{i+1} cannot be evaluated until the value of y_{i+1} inside the function $f(x_{i+1}, y_{i+1})$ is available. This value can be predicted using the Euler's formula as $y_{i+1} = y_i + h\,f(x_i, y_i)$.

Then Heun's formula becomes

$$\begin{aligned} y_{i+1} &= y_i + ((m_1 + m_2)/2)h \\ &= y_i + h/2(m_1 + m_2) \\ &= y_i + h/2\,[f(x_i, y_i) + f(x_{i+1}, y_{(i+1)})] \\ &= y_i + h/2\,[f(x_i, y_i) + f(x_{i+1}, y_i + h\,f(x_i, y_i))] \end{aligned}$$

where,

$$m_1 = f(x_i, y_i)$$
$$m_2 = f(x_{i+1}, y_{i+1})$$
$$y_{(i+1)} = y_i + h\,f(x_i, y_i)$$

Example 13.6: Given the equation $y'(x) = 2y/x$ with $y(1) = 2$. Estimate $y(2)$ using Heun's method , using $h = 0.25$.

Solution:

Iteration 1

$$m_1 = \frac{2\times 2}{1} = 4$$
$$y_e(1.25) = 2 + 0.25(4.0) = 3.0$$
$$m_2 = \frac{2\times 3.0}{1.25} = 4.8$$
$$y(1.25) = 2 + \frac{0.25}{2}(4.0 + 4.8) = 3.1$$

Iteration 2

$$m_1 = \frac{2\times 3.1}{1.25} = 4.96$$
$$y_e(1.5) = 3.1 + 0.25(4.96) = 4.34$$
$$m_2 = \frac{2\times 4.34}{1.5} = 5.79$$
$$y(1.5) = 3.1 + \frac{0.25}{2}(4.96 + 5.79) = 4.44$$

continue…..

Example:

$\frac{dy}{dx} = 3e^{-x} - 0.4y$ y(0) = 5 find y(3)for step size h = 1.5 using Heun's method.

Solution:

According to Huen's method,

$y_{i+1} = y_i + (m1+m2)/2 *h$

$m1 = f(x_i, y_i)$

$m_2 = f(x_{i+1}, y_{i+1})$

Iteration i = 0

Here, $y_0 = 5$ and $x_0 = 0$ and h=1.5

$m1 = f(x_0, y_0) = f(0, 5) = 3e^{-0} - 0.4(5) = 1$

$m2 = f(x_0 + h, y_0 + h) = f(0+1.5, 5+1*1.5)$; $y_0 + h = y_0 + m1*h$ (Euler's Method)

$= f(1.5, 6.5) = 3e^{-1.5} - 0.4(6.5) = - 1.9306$

$y_1 = y_0 + (m1+m2)/2 *h = 5+ (1 + (- 1.9306))/2 *1.5 = 4.302 \approx y(1.5)$

Iteration i = 1

Here, $y_1 = 4.302$ and $x_1 = 1.5$ and h=3

$m1 = f(x_1, y_1) = f(1.5, 4.302) = 3e^{-1.5} - 0.4(4.302) = - 1.0519$

$m2 = f(x_1 + h, y_1 + h) = f(1.5+1.5, 4.302+(- 1.0519)*1.5)$; $y_1 + h = y_1 + m1*h$

$= f(3, 2.726) = 3e^{-3} - 0.4(2.726) = - 0.9406$

$y_2 = y_1 + (m1+m2)/2 *h = 4.302+ ((- 1.0519) + - 0.9406))/2 *1.5 = 2.763 \approx y(3)$

Heun's method is a second order Runge-Kutta method because it employs slopes at two end points of the interval

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2. **Steven C.Chapra, Raymon P. Cannale.** *Numerical Methods for Engineers.* New Delhi : Tata McGRAW-HILL, 2003. ISMN 0-07-047437-0.